JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **34**, No. 1, February 2021
http://dx.doi.org/10.14403/jcms.2021.34.1.61

THE LANDAU-DE GENNES ENERGY IN A SMALL DOMAIN

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Abstract. In this paper, we investigate nontrivial equilibrium states of the Landau-de Gennes energy functional in a small domain.

1. Introduction

In this article, a system of liquid crystals occupying a domain in $\mathbf{R}^2$ governed by the Landau-de Gennes energy functional is considered. A liquid crystal is described by a symmetric and traceless 3×3 matrix $\mathbf{Q}$ which is given by

$$\mathbf{Q} = s_1(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}) + s_2(\mathbf{m} \otimes \mathbf{m} - \frac{1}{3}\mathbf{I})$$

where $\mathbf{u}$ and $\mathbf{m}$ are 3-dimensional unit vectors perpendicular to each other and s_i's are constants and $\mathbf{I}$ is the identity 3×3 matrix. We take a simplified version of the Landau-de Gennes energy as

$$E(\mathbf{Q}) = \int_\Omega f_L(\mathbf{Q}) + f_B(\mathbf{Q})\, dx$$

where

$$f_L(\mathbf{Q}) = \frac{1}{2}(L_1 Q_{ij,k}Q_{ij,k} + L_2 Q_{ij,j}Q_{ik,k} + L_3 Q_{ij,k}Q_{ik,j}),$$
$$f_B(\mathbf{Q}) = \frac{a}{2} tr\mathbf{Q}^2 - \frac{b}{3} tr\mathbf{Q}^3 + \frac{c}{4}(tr\mathbf{Q}^2)^2$$

Here, $Q_{ij,k} = \frac{\partial Q_{ij}}{\partial x_k}$(the partial derivative of the (i,j)-component Q_{ij} of $\mathbf{Q}$ with respect to x_k), L_i's are material dependent constants, and the

Received January 24, 2021; Accepted January 26, 2021.
2010 Mathematics Subject Classification: Primary 46T99, 34A34; Secondary 34K18, 49J99.
Key words and phrases: liquid crystal,bifurcation, nontrivial solution.
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This work was financially supported by the National Research Foundation of Korea with Grant No. 2017R1D1A1B06033070.

constants $a < 0$, $b > 0$, $c > 0$ are dependent of material and temperature. The energy density $f_L(\mathbf{Q}$ and $f_B(\mathbf{Q})$ are called the bulk and elastic energy respectively. For more details, we refer the reader to [2].

The problem we consider here is motivated by liquid crystal micro-droplets, so called tactoids, which spontaneously nucleate from isotropic dispersions followed by transforming into macroscopic anisotropic phases [5]. Such a phenomena appears in polymeric and colloidal liquid crystals. For simplicity, we take a two-dimensional liquid crystal tactoid governed by the Landau-de Gennes energy functional. Let Ω be a bounded domain containing the origin and

$$\Omega_r = \{rx | x \in \Omega\}.$$

Furthermore, we assume that the second order parameter $\mathbf{Q}$ takes a form

$$\mathbf{Q} = \begin{bmatrix} u_1 & u_2 \\ u_2 & -u_1 \end{bmatrix}.$$

Then the Landau-de Gennes energy functional is reduced to

$$E(\mathbf{u}) = \int_{\Omega_r} \frac{L}{2}|\nabla \mathbf{u}|^2 + (L_2 - L_3)\det(\nabla \mathbf{u}) + a|\mathbf{u}|^2 + c|\mathbf{u}|^4 \, dx.$$

where $L = 2L_1 + L_2 + L_3, \mathbf{u} = (u_1, u_2)$. The corresponding Euler-Lagrange equation with Dirichlet boundary condition $\mathbf{u} = 0$ on $\partial\Omega_r$ is given by

$$\begin{cases} -L\Delta \mathbf{u} + 2a\mathbf{u} + 4c|\mathbf{u}|^2\mathbf{u} = 0 & \text{in } \Omega_r, \\ \mathbf{u} = 0 & \text{on } \partial\Omega_r. \end{cases}$$

In the following section, we study existence of nontrivial solutions of the previous equation.

2. Existence of nontrivial solutions

We assume that the elastic constants L_i are very small so that $L_i << a, b, c$. Since the Landau-de Gennes energy functional is written as

$$\begin{aligned} E(\mathbf{u}) &= \int_{\Omega_r} \left\{ \frac{L}{2}|\nabla \mathbf{u}|^2 + c\left(|\mathbf{u}|^2 + \frac{a}{2c}\right)^2 - \frac{a^2}{4c} \right\} dx \\ &= \int_{\Omega_r} \left\{ \frac{L}{2}|\nabla \mathbf{u}|^2 + \frac{a^2}{4c}\left(\frac{2c}{-a}|\mathbf{u}|^2 - 1\right)^2 - \frac{a^2}{4c} \right\} dx, \end{aligned}$$

After scaling $\tilde{\mathbf{u}} = \sqrt{\frac{2c}{-a}}\mathbf{u}$, we replace $E(\mathbf{u})$ by

$$E(\mathbf{u}) = \int_{\Omega_r} \left\{ \frac{1}{2}|\nabla \mathbf{u}|^2 - \frac{a}{2L}\left(|\mathbf{u}|^2 - 1\right)^2 \right\} dx, \tag{2.1}$$

The Euler-Lagrange equation corresponding to (2.1) with $\mathbf{u} = 0$ on $\partial\Omega_r$ is given by

$$\begin{cases} -\Delta \mathbf{u} - \frac{2a}{L}\mathbf{u}(|\mathbf{u}|^2 - 1) = 0 & \text{in } \Omega_r, \\ \mathbf{u} = 0 & \text{on } \partial\Omega_r. \end{cases} \tag{2.2}$$

We denote $H_0^1(\Omega, \mathbf{R}^2)$ by X and let $Y = H^{-1}(\Omega, \mathbf{R}^2)$ be the dual space of $H_0^1(\Omega, \mathbf{R}^2)$. Define $J : \mathbf{R} \times X \to Y$ by

$$J(r, \mathbf{u})(\mathbf{v}) = \int_\Omega \left(-\Delta \mathbf{u} - \frac{2ar^2}{L}\mathbf{u}(|\mathbf{u}|^2 - 1) \right) \cdot \mathbf{v}\, dx$$

for $(r, \mathbf{u}) \in \mathbf{R} \times X$ and $\mathbf{v} \in X$. If $-\frac{2ar_0^2}{L}$ is a simple eigenvalue of $-\Delta$ with the Dirichlet boundary condition, then we have

$$J_{\mathbf{u}}(r_0, 0)(\mathbf{h}) : \mathbf{v} \to \int_\Omega \left(-\Delta \mathbf{h} + \frac{2ar_0^2}{L}\mathbf{h} \right) \cdot \mathbf{v}\, dx,$$

and $\ker(J_{\mathbf{u}}(r_0, 0)) = \{(\alpha, \beta)\phi \mid (\alpha, \beta) \in \mathbf{R}^2\}$, where

$$\begin{cases} -\Delta\phi = -\frac{2ar_0^2}{L}\phi \text{ in } \Omega \\ \phi = 0 \text{ on } \partial\Omega, \ \int_\Omega \phi^2 = 1. \end{cases} \tag{2.3}$$

Since $\dim\ker(J_{\mathbf{u}}(r_0, 0)) = 2$, we cannot apply Theorem 4.1 given in [1]. For each $(\alpha, \beta) \in \mathbf{R}^2$, we let $\Gamma_{(\alpha,\beta)} : \mathbf{R} \times H_0^1(\Omega, \mathbf{R}) \to H^{-1}(\Omega, \mathbf{R})$ be defined by

$$\Gamma_{(\alpha,\beta)}(r, u)(v) = \int_\Omega \left(-\Delta u - \frac{2ar^2}{L}\left((\alpha^2 + \beta^2)u^3 - u\right) \right) v\, dx$$

for all $v \in H_0^1(\Omega, \mathbf{R})$. It is immediate that

$$\left(\Gamma_{(\alpha,\beta)}\right)_u (r, 0)(h)(v) = \int_\Omega \left(-\Delta h + \frac{2ar^2}{L}h \right) v\, dx$$

for all $h, v \in H_0^1(\Omega, \mathbf{R})$.

Lemma 2.1. *If $-\frac{2ar_0^2}{L}$ is a simple eigenvalue of $-\Delta$ with the Dirichlet boundary condition, then we have*

(I) *dim* $\ker\left(\left(\Gamma_{(\alpha,\beta)}\right)_u (r_0, 0)\right) = 1$,

(II) $\mathcal{R} = $ *Range* $\left(\left(\Gamma_{(\alpha,\beta)}\right)_u (r_0, 0)\right)$ *is closed,*

(III) $\left(\Gamma_{(\alpha,\beta)}\right)_{u,r}(r_0,0)(\phi) \notin \mathcal{R}$.

Proof. Let $V = \ker\left(\left(\Gamma_{(\alpha,\beta)}\right)_u(r_0,0)\right)$ and $\mathcal{R} = \text{Range}\left(\left(\Gamma_{(\alpha,\beta)}\right)_u(r_0,0)\right)$. Then $V = \text{span}\{\phi\}$ and thus $\dim(V) = 1$. Since $H_0^1(\Omega,\mathbf{R})$ is Hilbert space, $H_0^1(\Omega,\mathbf{R}) = \text{span}\{\phi\} \oplus W$ where $W = (\text{span}\{\phi\})^\perp$. Hence W is closed. We see that $\mathcal{R} = \left(\Gamma_{(\alpha,\beta)}\right)_u(r_0,0)(W)$ and $\mathcal{R}$ is closed.

It follows from (2) that for any $w \in W = V^\perp$,

$$\int_\Omega \frac{-2ar_0^2}{L} w\phi \, dx = \int_\Omega w\left(\frac{-2ar_0^2}{L}\phi\right) dx = \int_\Omega w(-\Delta\phi)\, dx$$
$$= \int_\Omega \nabla w \cdot \nabla\phi \, dx = < w,\phi >_{H_0^1(\Omega,\mathbf{R})} = 0.$$

Since $(-\Delta)^{-1}$ is a compact operator [3], by Fredholm alternative [1] we have that for any $w \in H_0^1(\Omega,\mathbf{R}), < w,\phi >_{H_0^1(\Omega,\mathbb{R})} = 0$, the problem

$$\begin{cases} -\Delta h + \frac{2ar_0^2}{L}h = -\frac{2ar_0^2}{L}w & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

has a weak solution $h \in H_0^1(\Omega,\mathbf{R})$.

Now, we are in a position to show that $\text{codim}(\mathcal{R}) = 1$. Let $T \in H^{-1}(\Omega,\mathbf{R})$ be given. By Riesz's representation theorem, there exists $t \in H_0^1(B,\mathbf{R})$ such that

$$T(v) = < t,v >_{H_0^1(\Omega,\mathbf{R})} = \int_\Omega \nabla t \cdot \nabla v \, dx = \int_\Omega (-\Delta t)\, v \, dx$$

for all $v \in H_0^1(\Omega,\mathbf{R})$.

Since $t = \delta\phi + w_0$ for some $w_0 \in W, \delta \in \mathbf{R}$, we have

$$T(v) = \int_\Omega (-\Delta(\delta\phi + w_0))v \, dx = \delta \int_\Omega (-\Delta\phi)v \, dx + \int_\Omega (-\Delta w_0)v \, dx.$$

Since $w_0 \in W$, it follows from (2.4) that

$$\begin{cases} -\Delta h + \frac{2ar_0^2}{L}h = -\frac{2ar_0^2}{L}w_0 & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega \end{cases}$$

has a weak solution $h_0 \in H_0^1(\Omega,\mathbf{R})$. Let $s = h_0 + w_0$. Then we get

$$\int_\Omega (-\Delta w_0)v \, dx = \int_\Omega (-\Delta s + \frac{2ar_0^2}{L}s)v \, dx \text{ for all } v \in H_0^1(\Omega,\mathbf{R}).$$

We obtain that

$$\begin{aligned} T(v) &= \delta \int_\Omega (-\Delta\phi) v\, dx + \int_\Omega (-\Delta w) v\, dx. \\ &= \delta \int_\Omega (-\Delta\phi) v\, dx + \int_\Omega (-\Delta s + \frac{2ar_0^2}{L} s) v\, dx. \\ &= \delta\Phi(v) + \left(\Gamma_{(\alpha,\beta)}\right)(r_0, 0)(s)(v) \end{aligned}$$

where $\Phi \in H_0^1(\Omega, \mathbf{R})$ is defined by $\Phi(v) =< \phi, v >_{H_0^1(\Omega,\mathbf{R})}= \int_\Omega (-\Delta\phi) v\, dx$. Hence, $H^{-1}(\Omega, \mathbf{R}) = \text{span}\{\Phi\} \oplus \mathcal{R}$ and thus $\text{codim}(\mathcal{R}) = 1$.

In order to show (III), we fist note that for any $v \in H_0^1(\Omega, \mathbf{R})$, we get

$$\left(\Gamma_{(\alpha,\beta)}\right)_{u,r}(r_0, 0)(\phi)(v) = \int_\Omega \frac{4ar_0}{L} \phi v\, dx.$$

This implies that $\left(\Gamma_{(\alpha,\beta)}\right)_{u,r}(r_0, 0)(\phi) \in \text{span}\{\Phi\} = H^{-1}(\Omega, \mathbf{R}) \setminus \mathcal{R}$. □

For each $(\alpha, \beta) \in \mathbf{R}^2$, it follows from Theorem 4.1 in [1] that r_0 is a bifurcation point for $\Gamma_{(\alpha,\beta)}$. This enables us to conclude the following theorem.

THEOREM 2.2. *If $-\frac{2ar_0^2}{L}$ is a simple eigenvalue of $-\Delta$ with the Dirichlet boundary condition, then r_0 is a bifurcation point for J.*

Proof. For $(\alpha, \beta) \in \mathbb{R}^2$, we know that r_0 is a bifurcation point for $\Gamma_{(\alpha,\beta)}$. Let $\{(r_n, u_n)\}$ be a sequence such that

$$\begin{cases} \Gamma_{(\alpha,\beta)}(r_n, u_n) = 0, ||u_n|| \neq 0, \\ (r_n, u_n) \to (r_0, 0) \text{ as } n \to \infty. \end{cases}$$

Then for any $\mathbf{v} = (v_1, v_2) \in H_0^1(\Omega, \mathbf{R}^2)$, we get

$$\begin{aligned} 0 &= \alpha\Gamma_{(\alpha,\beta)}(r_n, u_n)(v_1) + \beta\Gamma_{(\alpha,\beta)}(r_n, u_n)(v_2) \\ &= \int_\Omega \left(-\Delta u_n - \frac{2ar_n^2}{L}((\alpha^2+\beta^2)u_n^2 - 1)u_n\right)(\alpha v_1 + \beta v_2)\, dx \\ &= \int_\Omega \left(-\Delta \mathbf{u}_n - \frac{2ar_n^2}{L}(|\mathbf{u}_n|^2 - 1)\mathbf{u}_n\right) \cdot \mathbf{v}\, dx \\ &= J(r_n, \mathbf{u}_n)(\mathbf{v}) \end{aligned}$$

where $\mathbf{u}_n = (\alpha, \beta)u_n$. Hence $(r_n, \mathbf{u}_n)$ be a sequence such that

$$\begin{cases} J(r_n, \mathbf{u}_n) = 0, \\ (r_n, \mathbf{u}_n) \to (r_0, 0) \text{ as } n \to \infty. \end{cases}$$

Therefore r_0 is bifurcation point for J. □

Corollary 2.3. *Suppose that there exists $r_0 \in \mathbf{R}$ such that $-\frac{2ar_0^2}{L}$ is simple eigenvalue of $-\Delta$ with the Dirichlet boundary condition. Then for r which is sufficiently close to r_0, there exists a nontrivial solution $\hat{\mathbf{u}}_r \in H_0^1(\Omega_r, \mathbf{R}^2)$ for (2.2).*

Proof. It follows from the previous theorem that for r which is sufficiently close to r_0, there exists a nontrivial weak solution $\mathbf{u}_r \in H_0^1(\Omega, \mathbf{R}^2)$ that is

$$J(r, \mathbf{u}_r)(\mathbf{v}) = 0 \text{ for any } \mathbf{v} \in H_0^1(\Omega, \mathbf{R}^2)$$

Then we have

$$\begin{cases} -\Delta \mathbf{u}_r - \frac{2ar^2}{L}\mathbf{u}_r(|\mathbf{u}_r|^2 - 1) = 0 & \text{in } \Omega, \\ \mathbf{u}_r = 0 & \text{on } \partial\Omega. \end{cases}$$

For any $\hat{x} \in \Omega_r$, let $\hat{\mathbf{u}}_r(\hat{x}) = \mathbf{u}_r(\hat{x}/r)$. Then $\hat{\mathbf{u}}_r$ satisfies (2.2). □

Remark 2.4. In fact, the solution $\hat{\mathbf{u}}_r$ found in Corollary 2.3 satisfies

$$\hat{\mathbf{u}}_r = \pm\sqrt{\frac{r - r_0}{e}}(\alpha, \beta)\phi + O(r - r_0) \text{ for some } e > 0,$$

where $\alpha^2 + \beta^2 \approx \frac{(r+r_0)e}{r^2 A}, A = \int_\Omega \phi^4 \, dx$. This can be proved by the Liapunov-Schmidt reduction[1, 4] and the arguments of chapter 5 in [1].

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